Composition series of affine manifolds and $n-$gerbes.

by

Tsemo Aristide.
The Abdus Salam center for theoretical Physics
Strada Costiera 11
34014 Trieste, Italy.
tsemo@ictp.trieste.it

Abstract.

In this paper, we study $n-$composition series of affine manifolds these are sequences $(M_n, \nabla_{M_n}) \rightarrow (M_{n-1}, \nabla_{M_{n-1}}) \rightarrow ...(M_1, \nabla_{M_1})$, where each affine map $f_i : (M_{i+1}, \nabla_{M_{i+1}}) \rightarrow (M_i, \nabla_{M_i})$ is surjective. One composition series are classified by gerbe theory. It is natural to think that $n-$composition series must be classified by $n-$gerbe theory. In the last section of this paper we propose a notion of abelian $n-$gerbe theory.

Introduction.

An affine bundle is a surjective affine map between affine manifolds. A composition serie of affine manifolds is a sequence $(M_n, \nabla_n) \rightarrow (M_{n-1}, \nabla_{M_{n-1}}) \rightarrow ... \rightarrow (M_2, \nabla_{M_2}) \rightarrow (M_1, \nabla_{M_1})$, where each map $f_i : (M_{i+1}, \nabla_{M_{i+1}}) \rightarrow (M_i, \nabla_{M_i})$ is an affine bundle.

When the source space M, of an affine bundle is compact, it becomes a locally trivial differentiable bundle by a well-known Ehresmann theorem (see [God] theorem 2.11 p.16). Let B and F be respectively the base and the fiber spaces of an affine bundle with compact total space. If moreover the second homotopy group of B is trivial, then by the Serre bundle theorem, one deduces that the first homotopy group $\pi_1(M)$ of M is an extension of $\pi_1(B)$ by $\pi_1(F)$. In particular this happens when (B, ∇_B) is geodesically complete.

Auslander has conjectured that the fundamental group of a compact geodesically complete affine manifold is polycyclic. The existence of a non trivial affine map, on a finite cyclic galoisian cover of a $n-$compact and complete affine manifold ($n > 2$) endowed with a complete structure eventually different from the pull-back, implies the Auslander conjecture [T4]. The classification of affine bundles whose total spaces are compact and complete and more generally of composition series of affine manifolds will conjecturally allow us to know all compact and complete affine manifolds, up to a finite cover, as we know the $2-$closed and complete affine manifolds.

The main goal of this paper is to study composition series of affine manifolds. First we study affine bundles.

Let $\pi_1(F)$ and $\pi_1(B)$ be two groups. Write $I\!R^{m+l} = I\!R^m \oplus I\!R^l$. We denote by $Aff(I\!R^m, I\!R^l)$ the group of affine maps of $I\!R^{m+l}$ which preserve $I\!R^l$, and by

$Aff_I(I\!R^m, I\!R^l)$ the subgroup of $Aff(I\!R^m, I\!R^l)$ whose restriction on $I\!R^m$ is the identity.

An algebra problem related to this classification problem of affine bundles is the following:

Given two representations $\pi_1(F) \to Aff(I\!R^l)$, and of $\pi_1(B) \to Aff(I\!R^m)$, classify all commutative diagrams:

$$\begin{array}{ccccccccc} 1 \to & \pi_1(F) & \to & \pi_1(M) \to & \pi_1(B) & \to 1 \\ & \downarrow & & \downarrow & \downarrow & \\ 1 \to & Aff_I(I\!R^m, I\!R^l) & \to & Aff(I\!R^m, I\!R^l) & \to Aff(I\!R^m) & \to 1 \end{array},$$

where the first line is an exact sequence.

There are many ways to solve the classification problem of affine bundles. First, we give a classification, of affinely locally trivial affine bundles, (see definition 2.2) after we solve the general case.

Let's present the classification of affinely locally trivial affine bundles. Let us consider an affinely locally trivial affine bundle with base (B, ∇_B) and typical fiber (F, ∇_F). We denote by T_F the group of translations of (F, ∇_F) (see section 3). The affine bundle f gives rise to representations $\pi_f : \pi_1(B) \to Aff(F, \nabla_F)/T_F$, $\pi'_f : \pi_1(B) \to Gl(T_F)$, and to a flat bundle $\hat{T}_F$, with typical fiber T_F over B associated to π'_f. We denote by T'_F the sheaf of affine sections of $\hat{T}_F$.

In [T5], we gave a classification of affinely locally trivial affine bundles using Hochschild cohomology classes for a representation of $\pi_1(B)$,

In this paper, we will give another using Cech cohomological classes via gerbe theory.

In fact it seems to us that the second classification fits best to our problem. Inspired by the philosophy of groupoid sheaves, we canonically associate to any representation $\pi : \pi_1(B) \to Aff(F, \nabla_F)/T_F$, a gerbe with lien T'_F which describes the gluing problem related to the existence of affinely locally trivial affine bundles associated to π. The $2-$cocycle described in Giraud's classification theorem of the associated gerbe, is given by an element of $H^2(B, T'_F)$. This class is the obstruction to the existence of affinely locally trivial affine bundles associated to π. When it vanishes, each element of $H^1(B, T'_F)$ defines an affinely locally trivial affine bundle. In this case the classification of the affinely locally trivial affine bundles is given by the orbits of elements of $H^1(B, T'_F)$ under a gauge group.

After the classification of affine bundles, we classify composition series of affine manifolds in which each map $f_i : (M_{i+1}, \nabla_{M_{i+1}}) \to (M_i, \nabla_{M_i})$ is an affinely locally trivial affine bundle. Since the classification of affinely locally trivial affine bundle has been done using gerbe theory, it is natural to think that the theory involved in the classification of composition series of affine manifolds is $n-$gerbe theory. In the last section of our work, we build a commutative $n-$gerbe theory.

This is the plan of our paper:

I. AFFINE BUNDLES.

1. Background.

An $n-$ connected affine manifold (M, ∇_M), is an $n-$ connected differentiable manifold endowed with a connection ∇_M, whose curvature and torsion forms vanish identically. The connection ∇_M defines on M an atlas (affine) whose transition functions are locally affine transformations of $I\!R^n$.

Let (M, ∇_M) and (N, ∇_N) be two affine manifolds respectively associated to the affine atlas, (U_i, ϕ_i) and (U'_j, ϕ'_j). An affine map between (M, ∇_M) and (N, ∇_N) is a differentiable map $f : M \to N$ such that $\phi' \circ f_{|U_i} \circ \phi_i^{-1}$ is an affine map. We denote by $App((M, \nabla_M), (N, \nabla_N))$ the set of affine maps between (M, ∇_M) and (N, ∇_N), and by $Aff(M, \nabla_M)$ the space of affine automorphisms of (M, ∇_M).

The affine structure of M pulls back to its universal cover $\hat{M}$, and defines on it an affine structure $(\hat{M}, \nabla_{\hat{M}})$, for which the universal cover map $p_M : \hat{M} \to M$ is an affine map. The affine structure of $(\hat{M}, \nabla_{\hat{M}})$ is defined by a local diffeomorphism $D_M : \hat{M} \to I\!R^n$ called the developing map.

The developing map gives rise to a representation $A_M : Aff(\hat{M}, \nabla_{\hat{M}}) \to Aff(I\!R^n)$ which makes the following diagram commute

$$\begin{array}{ccc} (\hat{M}, \hat{\nabla}_{\hat{M}}) & \xrightarrow{g} & (\hat{M}, \hat{\nabla}_{\hat{M}}) \\ \downarrow D_M & & \downarrow D_M \\ I\!R^n & \xrightarrow{A_M(g)} & I\!R^n \end{array}$$

where g is an element of $Aff(\hat{M}, \nabla_{\hat{M}})$. The restriction of A_M to the fundamental group $\pi_1(M)$ of M, is the holonomy representation h_M. The linear part $L(h_M)$ of h_M, is the linear holonomy of (M, ∇_M). It is in fact the holonomy of the connection ∇_M in the classical sense.

Definitions 1.1.

- The affine manifold (M, ∇_M) is complete, if and only if the developing map is a diffeomorphism. This is equivalent to saying that the connection ∇_M is geodesically complete.

- The affine manifold (M, ∇_M) is unimodular, if its linear holonomy lies in $Sl(n, \mathbb{R})$. Markus has conjectured that a compact affine manifold is complete if and only if it is unimodular.

- Let f and g be two affine bundles with the same base space (B, ∇_B), and respectively total spaces (M, ∇_M), and (N, ∇_N). An affine isomorphism between f and g, is an affine isomorphism between (M, ∇_M) and (N, ∇_N) which sends a fiber of f onto a fiber of g, and gives rise to an automorphism of (B, ∇_B).

2. Generalities.

This paragraph is devoted to some basic properties of affine bundles. In the sequel, we will suppose that all the fibers of a given affine bundle are diffeomorphic to each other.

Let $f : (M, \nabla_M) \to (M', \nabla_{M'})$ be an affine map, the map f pulls back to a map $\hat{f} : (\hat{M}, \nabla_{\hat{M}}) \to (\hat{M}', \nabla_{\hat{M}'})$ which makes the following diagram commute:

$$\begin{array}{ccc} (\hat{M}, \nabla_{\hat{M}}) & \xrightarrow{\hat{f}} & (\hat{M}', \nabla_{\hat{M}'}) \\ \downarrow p_M & & \downarrow p_{M'} \\ (M, \nabla_M) & \xrightarrow{f} & (M', \nabla_{M'}) \end{array} .$$

Proposition 2.1. [T4]. *Let (M, ∇_M) be the domain of an affine map. Suppose that M is compact. We denote by df_x, the differential df of f, at x. Then the distribution Df of M defined by*

$$Df_x = \{v \in T_x M / df_x(v) = 0\}$$

defines on M an affine bundle whose fibers are the leaves of the foliation defined by Df.

Sketch of proof.

As M is compact, the space of fibers is a differentiable manifold, say B. The transverse affine structure of the foliation Df, pushes forward to B and defines on it an affine connection ∇_B, which makes the projection $(M, \nabla_M) \to (B, \nabla_B)$ an affine map.

This proposition implies that an affine bundle with compact total space gives rise to a locally trivial differentiable bundle by a well-known Ehresmann result [God]. Denote by F the typical fiber. Applying the Serre bundle sequence to this bundle, we obtain the following short exact sequence:

$$\pi_2(B) \to \pi_1(F) \to \pi_1(M) \to \pi_1(B) \to 1.$$

If we suppose that $\pi_2(B) = 1$, then we obtain that $\pi_1(M)$ is an extension of $\pi_1(B)$ by $\pi_1(F)$. In particular this happens when (M, ∇_M) is complete. Remark that if (M, ∇_M) is complete, then the fibers and the base of the induced bundle are also complete.

Auslander has conjectured that the fundamental group of a compact and complete affine manifold is polycyclic. In [T4], we have conjectured that we

can change the complete affine structure of a galoisian cyclic finite cover of a $n-$compact, $(n > 2)$ and complete affine manifold to another complete one, so that it becomes the domain of a non trivial affine map. Non trivial means that the distribution Df is neither 0, nor the whole space. This conjecture implies the Auslander conjecture (see [T4]). As mentioned in the introduction, the classification of affine compact bundles will conjecturally allow us to know all compact and complete affine manifolds up to a finite cover.

In fact, there are examples of affine manifolds, which are total spaces of more than one non isomorphic affine bundle. The following is an example of this situation in dimension 3.

Let $C = (e_1, e_2, e_3)$ be a basis of $\mathbb{R}^3$. Consider the subgroup Γ of $Aff(\mathbb{R}^3)$, generated by f_1, f_2 and f_3, whose expressions in C are:

$f_1(x, y, z) = (x + 1, y, z)$

$f_2(x, y, z) = (x, y + 1, z)$

$f_3(x, y, z) = (x + y, y, z + 1)$.

The quotient of $\mathbb{R}^3$ by Γ is a compact affine manifold, M^3. The projections $p_2(x, y, z) = y$, and $p_3(x, y, z) = z$, define projections of M^3 over the circle endowed with its canonical complete structure. The bundles defined by those projections are not isomorphic. If they were isomorphic, there would exist an element of $Aff(\mathbb{R}^3)$ of the form $(x, y, z) \to (ax+b, f(y, z)+d(x))$ where f is an element of $Aff(\mathbb{R}^2)$, and d a linear map $\mathbb{R} \to \mathbb{R}^2$ which conjugates the map $(x, y, z) \to (x+1, y, z)$ to the map $(x, y, z) \to (x+1, y+z, z)$. This is evidently impossible. (For each bundle, we have adapted the expression of Γ in a basis (e'_1, e'_2, e'_3) such that the vector subspace $\mathbb{R}e'_1$ pulls forward on the base of each fibration).

Definition 2.2.

Let $f : (M, \nabla_M) \to (B, \nabla_B)$ be an affine bundle, We will say that the bundle f is an affinely locally trivial affine bundle, if and only if there exists an affine manifold (F, ∇_F) such that each element x of B, is contained in an open set U_x, such that there exists an affine isomorphism

$$f^{-1}(U_x) \to U_x \times (F, \nabla_F)$$

and the restriction of the projection on $f^{-1}(U_x)$ is the first projection $U_x \times (F, \nabla_F) \to U_x$ via this isomorphism.

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When the total space is compact, the last definition is equivalent to saying that one can build the Cech cocycle which defines the locally trivial differentiable structure of the affine bundle by affine maps.

In the previous examples the bundle defined by p_3 is affinely locally trivial, but not the one defined by p_2.

Let $f : (M, \nabla_M) \to (M', \nabla_{M'})$ be an affine map, where M and M' are respectively an n and an $n'-$manifold. The map f pulls back to a map $\hat{f} : (\hat{M}, \nabla_{\hat{M}}) \to (\hat{M}', \nabla_{\hat{M}'})$. There exists an affine map $f' : \mathbb{R}^n \to \mathbb{R}^{n'}$ which makes the following diagram commute:

$$\begin{array}{ccc} (\hat{M}, \nabla_{\hat{M}}) & \xrightarrow{\hat{f}} & (\hat{M}', \nabla_{\hat{M}'}) \\ \downarrow D_M & & \downarrow D_{M'} \\ I\!R^n & \xrightarrow{f'} & I\!R^{n'} \end{array}.$$

Let (M, ∇_M) be the total space of an affine bundle f. The foliation $\mathcal{F}_f$ defined by the leaves of f, pulls back to a foliation $\hat{\mathcal{F}}_f$ on $\hat{M}$, which is the pull-back of a foliation $D_M(\hat{\mathcal{F}}_f)$ of $I\!R^n$, by parallel $l-$affine subspaces, here n and l are the dimensions of M and of the fibers of the bundle.

Write $I\!R^n = I\!R^m \oplus I\!R^l$, where m is the dimension of the base of the bundle.

For every element γ of $\pi_1(M)$, we have:

$$h_M(\gamma)(x, y) = (A_\gamma(x) + a_\gamma, B_\gamma(y) + C_\gamma(x) + d_\gamma)$$

as $D_M(\hat{\mathcal{F}}_f)$ is stable under the holonomy.

An element γ of $\pi_1(M)$ which preserves a fiber of $\hat{f}$, preserves all the other fibers as the foliation $\mathcal{F}_f$ does not have holonomy. We obtain that

$$h_M(\gamma)(x, y) = (x, B_\gamma(y) + C_\gamma(x) + d_\gamma).$$

In fact we obtain a representation $\pi_1(F) \rightarrow Aff(I\!R^l)$ for each $x \in R^m$.

We deduce that, as they have been supposed to be diffeomorphic, all the fibers have the same linear holonomy.

The map

$$\pi_1(F) \longrightarrow I\!R^l$$

$$\gamma \longrightarrow C_\gamma(x) + d_\gamma$$

is a $1-$cocycle with respect to the linear holonomy. It defines the map

$$r : I\!R^m \rightarrow H^1(\pi_1(F), I\!R^l)$$

$$x \rightarrow [C_\gamma(x) + d_\gamma]$$

where $H^*(\pi_1(F), I\!R^l)$ is the $*$ cohomology group, with respect to the linear holonomy of the fibers. The cohomology class $r(x)$ is often called the radiance obstruction of the affine holonomy of the fiber over $p_B(x)$.

If the fibers are compact and complete, the image of r is contained in the algebraic subvariety L, of $H^1(\pi_1(F), I\!R^l)$ defined by

$$L = \{c \in H^1(\pi_1(F), I\!R^l) / \Lambda^l c \neq 0\}.$$

See [F-G-H] theorem 2.2.

The fact that f is an affinely locally trivial affine bundle, is equivalent to the fact that the map r is a constant map when the total space is complete.

Question. Are the fibers of an affine bundle isomorphic if its total space is compact ?

The following theorem was inspired by the last question:

Theorem 2.3. [T5]. *Suppose that the total space of an affine bundle is an $n-$compact and complete affine manifold, and moreover the fundamental group of the fibers are nilpotent. Then all the fibers are isomorphic to each other.*

Given an element γ of $\pi_1(B)$, and an element $x \in I\!R^m$, the restriction of the affine holonomy representation of $\pi_1(F)$ to $x \times I\!R^l$ and $(h_B(\gamma))(x)) \times I\!R^l$ are conjugated by an element of $Aff(I\!R^l)$, since they define the same affine structure (we choose $x \in D_B(\hat{B})$). This leads to define a gauge group for the linear representation $L(h_F)$.

Consider the subgroup G of automorphisms of $\pi_1(F)$ such that for every element g of G, there is a linear map B_g such that

$$L(h_F)(g(\gamma)) = B_g \circ L(h_F)(\gamma) \circ B_g^{-1}.$$

The group which elements are B_g will be called the gauge group of $L(h_F)$.

We associate to every $B_g \in G$ the following linear map of $H^p(\pi_1(F), I\!R^l)$: for each $c \in H^p(\pi_1(F), I\!R^l)$ we define $B_g^* c$ by

$$B_g^* c(\gamma_1, ..., \gamma_p) = B_g(c(g^{-1}(\gamma_1), ..., g^{-1}(\gamma_p))).$$

Two complete affine structures ∇_1 and ∇_2 on F, with same linear holonomy $L(h_F)$, and holonomy h_1 and h_2, are isomorphic if and only if there is an element B_g of the gauge group such that $B_g^* c_1 = c_2$, where c_1 and c_2 are respectively the radiance obstruction of h_1 and h_2.

3. Affinely locally trivial affine bundles.

Recall that, an affine bundle with complete total space is said to be an affinely locally trivial affine bundle, if and only if its pull-back of the bundle to the universal cover of the base is a trivial affine bundle.

In the sequel, (M, ∇_M) will be the compact total space of an affinely locally trivial affine bundle, with base space (B, ∇_B) and typical fiber (F, ∇_F).

The following proposition emphasizes the importance of the category of affinely locally trivial affine bundles.

Proposition 3.1. [T5] *Let f be an affine bundle whose total space is a complete affine $n-$manifold (not necessarily compact). If we suppose that the fibers are $2-$tori and moreover their linear holonomy is the linear holonomy of a complete structure of the $2-$torus distinct from the flat Riemmannian one, then f is an affinely locally trivial affine bundle.*

Let's go back to the classification problem.

Recall that $\pi_1(F)$ is a normal subgroup of $\pi_1(M)$. Let γ and γ_1 be respectively two elements of $\pi_1(F)$ and $\pi_1(M)$. We can write

$$h_M(\gamma) = (x, B_\gamma(y) + d_\gamma)$$

and

$$h_M(\gamma_1)(x, y) = (A_{\gamma_1}(x) + a_{\gamma_1}, B_{\gamma_1}(y) + C_{\gamma_1}(x) + d_{\gamma_1}),$$

where A_{γ_1} is an automorphism of $I\!R^m$, B_γ and B_{γ_1} are automorphisms of $I\!R^l$, $C_{\gamma_1} : I\!R^m \to I\!R^l$ is a linear map, a_{γ_1} is an element of $I\!R^m$ and d_γ, d_{γ_1} are elements of $I\!R^l$.

One has

$$h_M(\gamma_1)^{-1}(x,y) = (A_{\gamma_1}^{-1}(x) - A_{\gamma_1}^{-1}(a_{\gamma_1}), B_{\gamma_1}^{-1}(y) - B_{\gamma_1}^{-1}(d_{\gamma_1}) - B_{\gamma_1}^{-1}C_{\gamma_1}(A_{\gamma_1}^{-1}(x) - A_{\gamma_1}^{-1}(a_{\gamma_1}))).$$

Using the fact that $h_M(\pi_1(F))$ is a normal subgroup of $h_M(\pi_1(M))$, one obtains

$$\begin{array}{c} h_M(\gamma_1^{-1}) \circ h_M(\gamma) \circ h_M(\gamma_1) = \\ (x, B_{\gamma_1}^{-1}B_\gamma B_{\gamma_1}(y) + B_{\gamma_1}^{-1}B_\gamma C_{\gamma_1}(x) + B_{\gamma_1}^{-1}B_\gamma(d_{\gamma_1}) + B_{\gamma_1}^{-1}(d_\gamma) - B_{\gamma_1}^{-1}C_{\gamma_1}(x) - B_{\gamma_1}^{-1}(d_{\gamma_1})) \end{array}.$$

This implies that

$$B_{\gamma_1}^{-1}B_\gamma C_{\gamma_1}(x) - B_{\gamma_1}^{-1}C_{\gamma_1}(x) = 0;$$

we deduce that $C_{\gamma_1}(x) \in H^0(\pi_1(F), I\!R^l)$. The linear space $Applin(I\!R^m, H^0(\pi_1(F), I\!R^l))$ of linear maps $I\!R^m \to H^0(\pi_1(F), I\!R^l)$ has a left $\pi_1(B)-$module structure defined by

$$\gamma_1'(D) = B_{\gamma_1} \circ D$$

and a right $\pi_1(B)-$module structure defined by

$$\gamma_1'(D) = D \circ A_{\gamma_1},$$

where γ_1 is an element of $\pi_1(M)$ over an element γ_1' of $\pi_1(B)$.

We denote by T_F the connected component of the group of affine maps of (F, ∇_F), which pull-back on translations of $I\!R^l$. The linear map of $H^0(\pi_1(F), I\!R^l)$ defined by

$$t \to B_{\gamma_1} t$$

induces a linear map of T_F. This induces a $\pi_1(B)$ left structure on $Applin(I\!R^m, T_F)$. The right structure of $\pi_1(B)$ on $Applin(I\!R^m, H^0(\pi_1(F), I\!R^l))$ also induces a right $\pi_1(B)$ structure on $Applin(I\!R^m, T_F)$. Let $Z\!\!Z\pi_1(B)$ be the group algebra of the group $\pi_1(B)$. The vector space $Applin(I\!R^m, T_F)$ is endowed with a $Z\!\!Z\pi_1(B)$ Hochschild module structure.

The bundle is supposed to be affinely locally trivial; this implies that his lifts on $\hat{B}$, is $\hat{B} \times (F, \nabla_F)$. The action of $\pi_1(B)$ on $\hat{B} \times (F, \nabla_F)$ is made by affine maps. We deduce that $\pi_1(F)$ is normal in $\pi_1(M)$, and a representation $\pi_f : \pi_1(B) \to Aff(F, \nabla_F)/T_F$ induced by its action on $\hat{B} \times (F, \nabla_F)$.

Recall the following problem stated in [Bry]. Let

$$1 \to A \to G \to H \to 1$$

be an exact sequence of groups, where A is commutative. Given an $H-$bundle over the manifold X, we want to classify all the bundles over X, with structural group G, which are lifts of the previous bundle.

Our problem is quite similar: We have seen that an affinely locally trivial affine bundle f gives rise to a representation $\pi_f : \pi_1(B) \to Aff(F, \nabla_F)/T_F$. We denote by π'_f the flat bundle induced by π_f. Given such a representation $\pi : \pi_1(B) \to Aff(F, \nabla_F)/T_F$ with associated $Aff(F, \nabla_F)/T_F$ bundle π', we want to classify all affinely locally trivial affine bundles associated.

To make our theory fit in gerbe theory with the band T'_F, the sheaf of affine section of $\hat{T}_F$, we will consider first, the classification of affine bundles up to T'_F isomorphisms.

Now let's recall some general facts of sheaf of groupoids and descent theory. Our exposition follows the treatment of [Bry]. The general philosophy is to express gluing conditions in terms of covering maps. We remark that if a manifold is an affine manifold, so are its covers, the category of affine manifolds is stable under affine fiber product.

For every locally isomorphic affine map $g : (Y, \nabla_Y) \to (B, \nabla_B)$, we can pull-back the affinely locally trivial affine bundle π' over (B, ∇_B) in a bundle π'_Y over (Y, ∇_Y) (associated to π_Y). The total space of the pull-back is $Q = \pi' \times_B (Y, \nabla_Y)$ and the fiber map the canonical projection.

Consider $Y \times_B Y$ with the two canonical projections $p_1, p_2 : Y \times_B Y \to Y$. We denote by $p_i^* Q$ $i = \{1, 2\}$ the pull-back of Q by p_i. There is a natural isomorphism $\phi : p_1^* Q \to p_2^* Q$. This isomorphism satisfies the following cocycle condition

$$p_{13}^*(\phi) = p_{23}^*(\phi) \circ p_{12}^*(\phi), \tag{2}$$

an equality of morphisms $p_1^* Q \to p_3^* Q$ of affine bundles over $Y \times_B Y \times_B Y$, where p_1, p_2 and p_3 are the canonical projections on the three factors, and p_{12}, p_{13} and p_{23} are the canonical projections of $Y \times_B Y \times_B Y$ over $Y \times_B Y$.

Conversely, given an affine bundle $Q \to Y$ which satisfies the condition (2), we recover an affine bundle over (B, ∇_B).

In fact one obtains:

Proposition 3.2. *Let $g : (Y, \nabla_Y) \to (B, \nabla_B)$ be a local isomorphism of affine manifolds. The pull-back functor g^* induces an equivalence of categories between the category of affine bundles over (B, ∇_B) and the category of affine bundle bundles over (Y, ∇_Y) equipped with a descent isomorphism $\phi : p_1^* Q \to p_2^* Q$ satisfying the cocycle condition (2).*

The general definition of torsor adapted to this case is:

Définition 3.3. *A T'_F torsor, will be a sheaf H on (B, ∇_B), together with a T'_F action such that every point of B has a neighborhood U with the property that for every $V \subset U$ open, the space $H(V)$ is an affine principal bundle with structural group $T'_{F|V}$.*

The isomorphism classes of T'_F torsors are given by $H^1(B, T'_F)$, where $H^*(B, T'_F)$ is the $*$ cohomology group of T'_F related to the usual Cech cohomology.

As the sheaf T'_F is a locally constant sheaf, the notion of T'_F torsor in our case is similar to a notion of affine T'_F bundle over (B, ∇_B).

We can associate to a representation $\pi : \pi_1(B) \to Aff(F, \nabla_F)/T_F$, the following sheaf of groupoids, B_π. To every local affine isomorphism $(Y, \nabla_Y) \to (B, \nabla_B)$, we associate the category Y_π, whose objects are affinely locally trivial affine bundles over (Y, ∇_Y) with typical fiber (F, ∇_F), associated to π_Y. The (auto)morphisms are T'_F-automorphisms.

It is easy to show that the following properties are satisfied:

(i) For every diagram $(Z, \nabla_Z) \xrightarrow{g} (Y, \nabla_Y) \xrightarrow{h} (B, \nabla_B)$ of local affine isomorphisms, there is a functor $g^{-1} : Y_\pi \to Z_\pi$;

(ii) For every diagram $(W, \nabla_W) \xrightarrow{k} (Z, \nabla_Z) \xrightarrow{g} (Y, \nabla_Y) \xrightarrow{h} (B, \nabla_B)$ there is an invertible natural transformation $\theta_{g,k} : k^{-1}g^{-1} \to (gk)^{-1}$.

This makes our category a presheaf category. Moreover properties a la (2) are satisfied to ensure that some kind of Haefliger $1-$cocycles is satisfied, in order to make our presheaf of category a sheaf of category.

One can more usually define a sheaf of category. It is a map C on the family of open subsets of B

$$U \longrightarrow C(U)$$

which assigns to any open subset U of B a category $C(U)$.

For every open subset $V \subset U$, there is a composition of morphisms from $C(U)$ to $C(V)$. When $U = V$, this composition is just the compostion of morphisms. This defines the presheaf of category. Moreover a descent condition is needed to make the presheaf a sheaf.

In fact, our sheaf of category is a gerbe with band T'_F. It means that the following properties are satisfied

(G1) Given any object of Y_π, the sheaf of local automorphisms of this object is a sheaf of groups which is locally isomorphic to T'_F.

(G2) Given two objects Q_1 and Q_2 of Y_π, there exists a local isomorphism surjective map $g : Z \to Y$ such that $g^{-1}Q_1$ and $g^{-1}Q_2$ are locally isomorphic.

(G3) There is a local isomorphism surjective affine map $Y \to X$ such that the category Y_π is not empty.

One say that our sheaf of category is a gerbe with band or lien T'_F.

Remark.

To ensure the axiom (G3) to be satisfied, one may show an affinely local trivial affine bundle with typical fiber (F, ∇_F), over $\hat{B}$. This bundle is just the trivial one.

Let's now state the first classification theorem which is an adaptation of the Giraud classification theorem.

Theorem 3.4. *The set of equivalence classes of the gerbes is in one to one correspondence with $H^2(B, T'_F)$.*

Proof.

Let's consider a cover $(U_i)_{i\in I}$ of B by open $1-$connected affine charts. The T'_F-automorphisms of an object P_i of $C(U_i)$, is isomorphic to the restriction of T'_F to U_i.

There is a T'_F-isomorphism

$$u_{ij}:(P_i)_{|C(U_{ij})}\to(P_j)_{|C(U_{ij})}$$

in the category $C(U_{ij})$. We define a section h_{ijk} of T'_F by,

$$h_{ijk}=u_{ik}^{-1}u_{ij}u_{jk}\in Aut(P_k)\simeq T'_F.$$

In fact $h=(h_{ijk})$ is a T'_F-valued Cech $2-$cocycle. The corresponding class in $H^2(B,T'_F)$ is independent of all the choices. We will show that this correspondence defines an isomorphism between the group of equivalence classes, and the set of isomorphic gerbes with band T'_F.

To show the injectivity of this map, one remarks that if the cohomology class defined by h is trivial, then one can modify the isomorphisms u_{ij} such that $u_{ik}=u_{ij}u_{jk}$. We then obtain an affine T'_F torsor over (B,∇_B) which represents a trivial gerbe.

To prove the surjectivity, we construct a gerbe associated to a $2-$Cech cocycle $h=(h_{ijk})$ with values in T'_F. It is sufficient to find a family of elements u_{ij} of T'_F-automorphisms of U_{ij} such that the condition $u_{ik}^{-1}u_{ij}u_{jk}=h_{ijk}$ is satisfied.

This is our classification theorem for affinely locally trivial affine bundles, up to T'_F-isomorphisms.

Theorem 3.5. *Let $\pi:\pi_1(B)\to Aff(F,\nabla_F)/T_F$ be a representation. Then there are affine bundles over π', if and only if its associated gerbe is trivial. In this case the T'_F-isomorphism classes of affine bundles are given by $H^1(B,T'_F)$.*

Proof.

If there exists an affine bundle associated to the representation π, the boundary of the cocycle which defines the fibration represents the associated gerbe, so this gerbe is trivial.

On the other hand, the $2-$cocycle associated can be described as follows:

Consider a trivialisation of the flat bundle π', associated to π.

For every i,j such that $U_i\cap U_j\neq\emptyset$, we have

$$U_i\cap U_j\times Aff(F,\nabla_F)/T_F\longrightarrow U_i\cap U_j\times Aff(F,\nabla_F)/T_F$$

$$(x,y)\longrightarrow(x,g'_{ij}(y)).$$

We denote by $g_{ij}(x)$, an element of $Aff(F,\nabla_F)$ over g'_{ij} which depends affinely on x. We set

$$h_{ijk}=g_{ik}^{-1}g_{ij}g_{jk}.$$

We have seen that if the cocycle is trivial, one can find a family of maps $w_{ij}:U_i\cap U_j\to T'_F$ such that

$$(g_{ik}+w_{ik})=(g_{ij}+w_{ij})(g_{jk}+w_{jk}).$$

Consider the family of maps

$$\phi_{ij} : U_i \cap U_j \times (F, \nabla_F) \longrightarrow U_i \cap U_j \times (F, \nabla_F)$$

$$(x, y) \longrightarrow (x, (g_{ij} + w_{ij}(x))(y))$$

We have

$$\phi_{ij} \circ \phi_{jk}(x, y) = (x, (g_{ij} g_{jk} + w_{ij}(x) + g_i w_{jk}(x))(y)).$$

One sees that the Cech cocycle condition is verified.
For every other $1-$cocycle (v_{ij}), one gets an affine bundle by setting

$$\phi'_{ij} : U_i \cap U_j \times (F, \nabla_F) \longrightarrow U_i \cap U_j \times (F, \nabla_F)$$

$$(x, y) \longrightarrow (x, (g_i + (w_{ij} + v_{ij})(x))(y)).$$

Two different cocycles used to define affinely locally trivial affine bundles can define isomorphic affine bundles.

Let f_1 and f_2 be two isomorphic affinely locally trivial affine bundles whose total spaces are $n-$compact, and which induce the same sheaf T'_F. There is an affine transformation g of $\hat{f}_1$ which preserves the foliation $\hat{\mathcal{F}}_{f_1}$ (where $\hat{\mathcal{F}}_{f_1}$ is the pull-back of the the foliation induced by f_1) and conjugates the Deck transformations which defines the total space of f_1, in those which define the one of f_2. As both bundles induce T'_F, their induced represions $\pi_1(B) \rightarrow Aff(F, \nabla_F)/T_F$ coincide. The cohomology class defined by f_1 is changed in the class defined by f_2 by g, one has

Proposition 3.6. *The isomorphim classes of affine bundles are given by the quotient of $H^1(B, T'_F)$ by the action of a gauge group. This group is the group of affine automorphism of $\hat{f}_1$ which preserve its fibers, are pulls back of automorphisms of (B, ∇_B) and give rise to the same bundle π'.*

4. The general case.

In the previous section of our paper, we have classified affinely locally trivial affine bundles. In the following section, we consider the more general situation, when the total space is supposed to be only compact and complete.

Given an affine bundle with compact and complete total space say (M, ∇_M) and base space (B, ∇_B), we have seen that the fibers inherit affine structures from the total space which are not necessarily isomorphic, but which have the same linear holonomy. Let γ be an element of $\pi_1(M)$, set $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^l$ where n, m, and l are respectively the dimension of M, B and the typical fiber F. We have:

$$h_M(\gamma)(x, y) = (A_\gamma(x) + a_\gamma, B_\gamma(y) + C_\gamma(x) + d_\gamma),$$

where A_γ and B_γ are respectively affine automorphisms of $\mathbb{R}^m$ and $\mathbb{R}^l$, a_γ and d_γ are respectively elements of $\mathbb{R}^m$ and $\mathbb{R}^l$, and $C : \mathbb{R}^m \rightarrow \mathbb{R}^l$ is a linear map.

If γ lies in $\pi_1(F)$, then $A_\gamma = I_{\mathbb{R}^m}$ and $a_\gamma = 0$.

Now consider an $m-$affine manifold (B, ∇_B) compact and complete, a compact $l-$manifold F and a representation $L(h_F) : \pi_1(F) \to Gl(l, \mathbb{R})$ which is the linear holonomy of a complete affine structure of F. We want to classify all affine bundles with complete total space with base (B, ∇_B) and whose fibers are diffeomorphic to F and inherit affine structures from the total space whose linear holonomy is $L(h_F)$.

The natural question which arises is: Does every map $r : \mathbb{R}^m \to H^1(\pi_1(F), \mathbb{R}^l)$ give rise to an affine bundle ?

For every element γ_1 of $\pi_1(B)$, the affine representations defined by the cocycles $r(x)$ and $r(\gamma_1(x))$ must be isomorphic.

Recall that we have defined a gauge group G of the representation $L(h_F)$ as follows: it is a group of linear maps such that for every element $B_g \in G$ there is an automorphism g of $\pi_1(F)$ which satisfies

$$L(h_F)(g(\gamma)) = B_g L(h_F)(\gamma) B_g^{-1}.$$

The group G acts on $L(h_F)$ one cocycles by setting

$$(B_g^* c)(\gamma) = B_g c(g^{-1}(\gamma)).$$

Let γ_1 and γ be respectively elements of $\pi_1(M)$ and $\pi_1(F)$. We have

$$\begin{gathered}\gamma_1 \circ \gamma \circ \gamma_1^{-1}(x, y) = \\ (x, B_{\gamma_1} B_\gamma B_{\gamma_1}^{-1}(y) - B_{\gamma_1} B_\gamma B_{\gamma_1}^{-1} C_{\gamma_1}(A_{\gamma_1}^{-1}(x) - A_{\gamma_1}^{-1}(a_{\gamma_1})) - B_{\gamma_1} B_\gamma B_{\gamma_1}^{-1}(d_{\gamma_1}) + \\ B_{\gamma_1} C_\gamma (A_{\gamma_1}^{-1}(x) - A_{\gamma_1}^{-1}(a_{\gamma_1})) + B_{\gamma_1}(d_\gamma) + C_{\gamma_1}(A_{\gamma_1}^{-1}(x) - A_{\gamma_1}^{-1}(a_{\gamma_1})) + d_{\gamma_1})\end{gathered}.$$

The map

$$i'(\gamma_1) : \pi_1(F) \longrightarrow \pi_1(F)$$

$$\gamma \longrightarrow \gamma_1 \gamma \gamma_1^{-1}$$

is an automorphism associated to the element of the gauge group B_{γ_1}. If γ_1 lies in $\pi_1(F)$ the induced map on $H^1(\pi_1(F), \mathbb{R}^l)$ is trivial. We deduce a map

$$i : \pi_1(B) \longrightarrow Gl(H^1(\pi_1(F), \mathbb{R}^l))$$

$$\gamma_1 \longrightarrow ai'(\gamma_1'),$$

where γ_1 pulls back to γ_1' and $ai'(\gamma_1')$ is the action of γ_1' on $H^1(\pi_1(F), \mathbb{R}^l)$ induced by $B_{\gamma_1'}$.

We have $r(A_{\gamma_1'}(x) + a_{\gamma_1'}) = [B_{\gamma_1'} r(x)]$.

It follows that the following square is commutative

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{\gamma_1} & \mathbb{R}^m \\ \downarrow r & & \downarrow r \\ H^1(\pi_1(F), \mathbb{R}^l) & \xrightarrow{i(\gamma_1)} & H^1(\pi_1(F), \mathbb{R}^l) \end{array}.$$

The representation i allows also to construct a bundle over (B, ∇_B) with typical fiber $H^1(\pi_1(F), I\!R^l)$. We also denote by i this bundle. The map r can also be viewed as a section of this bundle.

We will classify all affine bundles for a given representation i, and a map r such that each $r(x)$ defines an affine structure.

The map r defines a representation $\pi_1(F) \longrightarrow Aff(I\!R^n)$ such that the quotient $I\!R^n/\pi_1(F)$ is an affine bundle over $I\!R^m$. We assume that for each $\gamma \in \pi_1(B)$, there is an element of $Aff(I\!R^n/\pi_1(F))$ which induces $i(\gamma)$.

Let U be an open set of (B, ∇_B). We define the following sheaf of categories

$$U \longrightarrow C(U).$$

Where $C(U)$ is the set of affine bundles such that the canonical bundle with typical fiber $H^1(\pi_1(F), I\!R^l)$ associated, is the restriction of i to U, and whose lifts on the universal cover of U is the restriction of $I\!R^n/\pi_1(F)$ to it.

The sheaf of categories C is a gerbe with band $Aff(I\!R^n/\pi_1(F))_0$, which denotes the sheaf induced by the connected component of the affine automorphisms of $I\!R^n/\pi_1(F)$ which pushes forward on the identity of $I\!R^m$.

Let us explain why the band is $Aff(I\!R^n/\pi_1(F))_0$.

Consider a trivialization of i,

$$\bar{h}_{ij} : U_i \cap U_j \times H^1(\pi_1(F), I\!R^l) \longrightarrow U_i \cap U_j \times H^1(\pi_1(F), I\!R^l)$$

$$(x, y) \longrightarrow (x, \bar{h}_{ij}(y)).$$

Here the U_i are connected open sets of affine charts. So we can restrict the bundle $I\!R^n/\pi_1(F)$ to each U_i. We denote by U_i^F this restriction.

To $\bar{h}_{ij}$ we associate an element h_{ij} of $Aff(U_i^F)$ which pushes forward on the identity of U_i, and gives rise to $\bar{h}_{ij}$. The maps

$$h_{ik}^{-1} h_{ij} h_{jk}$$

are the obstructions of the existence of an affine bundle associated to i and r.

Proposition 4.1. *The map $h_{ijk} = h_{ik}^{-1} h_{ij} h_{jk}$ is an element of the restriction of $Aff(I\!R^n/\pi_1(F))_0$ to $U_i \cap U_j \cap U_k$.*

Proof.

We deduce this, from the fact that, the map h_{ijk} gives rise to the identity of $H^1(\pi_1(F), I\!R^l)$, since it is shown in [T2] that the set of affine automorphisms which commutes with the holonomy of a compact and complete affine manifold is a cover of the connected component of its affine automorphism group.

This is our classification theorem in the general case.

Theorem 4.2. *For each representation i and map r, there is a $2-$Cech cocycle which is the obstruction of the existence of an affine bundle associated to i and r. When it vanishes the set of isomorphisms classes of affine bundles*

are given by the orbit of elememt of $H^1(B, Aff(I\!R^n/\pi_1(F))_0)$ under a gauge group.

II. COMPOSITION SERIES OF AFFINE MANIFOLDS.

Recall that a $n-$affine manifold is said to be complete if and only if the connection ∇_M is complete. This is equivalent to saying that M is the quotient of $I\!R^n$ by a group Γ_M of affine automorphisms which act properly and freely on $I\!R^n$. In this part, we will only consider complete affine manifolds.

The representation $h_M : \Gamma_M = \pi_1(M) \to Aff(I\!R^n)$ is called the holonomy of the affine manifold (M, ∇_M). Its linear part $L(h_M)$ is called the linear holonomy.

It has been conjectured by Auslander that the fundamental group of a compact and complete affine manifold is polycyclic.

In [T4], we have conjectured that each compact and complete $n > 2$ affine manifold (M, ∇_M) has a finite cyclic and galoisian cover M' endowed with a complete affine structure $(M', \nabla'_{M'})$ eventually different from the pull back such that $(M', \nabla_{M'})$ is the source space of a non trivial affine map. Non trivial means that the fibers of f are neither M' nor points of M'. This conjecture implies the Auslander conjecture.

If we suppose that the source space (M, ∇_M), of a non trivial affine map is compact, then (M, ∇_M) is the source space of a non trivial affine surjection f over a manifold (B, ∇_B). We deduce from a well-known Ehresmann theorem, that f is also a locally trivial differentiable fibration. All the fibers of an affine fibration inherit affine structures from (M, ∇_M) with same linear holonomy.

The last conjecture leads to the following problem: Given two affine manifolds (B, ∇_B) and (F, ∇_F) classify every affine surjection $f : (M, \nabla_M) \to (B, \nabla_B)$ such that the differentiable structure of the fibers is F and their linear holonomy is the one of (F, ∇_F). Or more generally, Given n affine manifolds (F_i, ∇_{F_i}), classify all composition series $(M_{n+1}, \nabla_{M_{n+1}}) \to (M_n, \nabla_{M_n}) \to ... \to (M_1, \nabla_{M_1})$ such that every map $f_i : (M_{i+1}, \nabla_{M_{i+1}}) \to (M_i, \nabla_{M_i})$ in the last sequence is an affine surjection with fiber diffeomorphic to F_i, and which inherits from $(M_{i+1}, \nabla_{M_{i+1}})$ an affine structure wich linear holonomy is the linear holonomy of (F_i, ∇_{F_i}).

We have classify in the first part affine surjections with compact total spaces using gerbe theory. The purpose of this part is to classify affine compostion series of affine manifolds. We restrict to composition series $(M_n, \nabla_{M_n}) \to (M_{n-1}, \nabla_{M_{n-1}}) \to ... \to (M_1, \nabla_{M_1})$ such that the projection $f_i : (M_{i+1}, \nabla_{M_{i+1}}) \to (M_i, \nabla_{M_i})$ is an affinely locally trivial affine (a.l.t) fibration. This means that the holonomy of the fibers is fixed.

Two composition series

$$({M_n}^j, \nabla_{{M_n}^j}) \to (M^j_{n-1}, \nabla_{M_{{n-1}^j}}) \to ...({M_2}^j, \nabla_{{M_2}^j}) \to (M_1, \nabla_{M_1}), j = 1, 2$$

are equivalent if and only if the bundle ${f_i}^1$ and ${f_i}^2$ are isomorphic in respect to T_{F_i} isomorphisms. This means that we consider isomorphisms of affine fibrations which act by translations on the fibers and project on the identity of the base space.

The classification of affinely locally trivial affine bundles where made using commutative gerbe theory. It is natural to think that the classification of affinely locally $n-$ composition series must be done using $n-$gerbe theory. In this part, we will give a classification of $n-$ composition series of affine manifolds, and after conceptualize the ideas involved to give a theory of commutative $n-$gerbes.

1. 3 compostion series of affine manifolds.

Let's recall first the classification of affinely locally trivial affine bundles up to translational isomorphisms with given fiber and base space.

We have two affine manifolds (B, ∇_B) and (F, ∇_F) which represent respectively the base space and the fiber of the affine bundles we intend to classify.

The structure of a locally trivial affine bundle gives rise to a map $\pi_{BF} : \pi_1(B) \rightarrow Aff(F, \nabla_F)/T_F$ which defines a locally flat bundle PB_F over B with typical fiber $Aff(F, \nabla_F)/T_F$. This principal bundle induces a flat bundle BT_F over B with typical T_F.

Let T'_F be the sheaf of affine sections of BT_F. We define a commutative gerbe C with lien T'_F as follows:

To each open set U of B, we associate the category $C(U)$ of affinely locally trivial affine bundles with typical fiber (F, ∇_F) such that the canonical UT_F bundle associated to it, is the restriction of BT_F to U.

To represent the classifying two cocycle associated to C, we consider an open covering U_k of B by connected affine charts, in each category $C(U_k)$ we choose an objet which is an affine bundle isomorphic to $U_k \times (F, \nabla_F)$.

The trivialization of the bundle PB_F gives rise to:

$$U_k \cap U_l \times Aff(F, \nabla_F)/T_F \longrightarrow U_k \cap U_l \times Aff(F, \nabla_F)/T_F$$

$$(x, y) \longmapsto (x, \bar{t}_{kl}(y)),$$

where $\bar{t}_{kl}$ is an element of $Aff(F, \nabla_F)/T_F$. Taking for each k, l a map t_{kl} over $\bar{t}_{kl}$ which depends affinely of x, we obtain:

$$U_k \cap U_l \times (F, \nabla_F) \longrightarrow U_k \cap U_l \times (F, \nabla_F)$$
$$(x, y) \longmapsto (x, t_{kl}(x)y)$$

Then we have the family of maps

$$t_{klm} : U_k \cap U_l \cap U_m \longrightarrow T_F$$

$$x \longmapsto t_{kl} t_{lm} t_{km}^{-1}$$

This family of maps t_{klm} is a $2-$cocycle which classifies the gerbe C. It is the obstruction to the existence of a locally trivial affine bundle over (B, ∇_B) associated to BT_F. In this case, the set of isomorphic classes of translational affine bundles (or the classes of T_F isomorphic bundles) with typical fiber (F, ∇_F) and base space (B, ∇_B) are given by the Cech cohomology group $H^1(B, T'_F)$ of the sheaf T'_F.

Remark.

Consider the composition serie $(M_3, \nabla_{M_3}) \to (M_2, \nabla_{M_2}) \to (M_1, \nabla_{M_1})$. The fact that the affine bundles $(M_3, \nabla_{M_3}) \to (M_2, \nabla_{M_2})$ and the affine bundle $(M_2, \nabla_{M_2}) \to (M_1, \nabla_{M_1})$ are affinely locally trivial affine bundles does not imply that the bundle $(M_3, \nabla_{M_3}) \to (M_1, \nabla_{M_1})$ is a locally trivial affine bundle.

This can be illustrated by the following example. Consider the subgroup Γ of $\mathbb{R}^3$ generated by the three maps f_1, f_2 and f_3 defined by

$$f_1(x,y,z) = (x+1, y, z)$$

$$f_2(x,y,z) = (x, y+1, z)$$

$$f_3(x,y,z) = (x+y, y, z+1).$$

In the canonical basis (e_1, e_2, e_3) of $\mathbb{R}^3$. The quotient of $\mathbb{R}^3$ by Γ is a three compact affine manifold M^3. The projection p_1 of $\mathbb{R}^3$ on its subvector space V_2 generated by e_2 and e_3 parallel to the one generated by e_1, defines an affinely locally trivial affine bundle over the torus T_2 endowed with its canonical riemannian flat structure.

The projection p_2 of V_2 on the line generated by e_2 parallel to the one generated by e_3 defines an affinely locally trivial affine bundle of the torus over the circle.

It is easy to see that the projection $p_2 \circ p_1$ defines an affine bundle which is not an affinely locally trivial affine bundle over the circle.

Before to go to the general case, we will treat $3-$ series of composition. So we have a sequence $(M_3, \nabla_{M_3}) \to (M_2, \nabla_{M_2}) \to (M_1, \nabla_{M_1})$ of affine maps such that $f_{i-1} : (M_i, \nabla_{M_i}) \to (M_{i-1}, \nabla_{M_{i-1}})$ defines an affinely locally trivial affine bundle.

We have supposed that the total spaces of our bundles are compact and complete affine manifolds. This implies that $\pi_1(F_2)$ is normal in $\pi_1(M_3)$ and $\pi_1(F_1)$ is normal in $\pi_1(M_2)$, thus we have the following exact sequences

$$1 \to \pi_1(F_1) \to \pi_1(M_2) \to \pi_1(M_1) \to 1,$$

and

$$1 \to \pi_1(F_2) \to \pi_1(M_3) \to \pi_1(M_2) \to 1.$$

We denote by $n_i, i = 1,2,3$ the dimensions of M_i, and by $l_i, i = 1,2$ the dimensions of F_i. We put $\mathbb{R}^{n_3} = \mathbb{R}^{n_1} \oplus \mathbb{R}^{l_1} \oplus \mathbb{R}^{l_2}$,

Let γ be an element of $\pi_1(M_3)$; the (M_3, ∇_{M_3}) holonomy's action of γ on $\mathbb{R}^{n_3}$ is given by

$$\gamma(x,y,z) = (A_1^\gamma(x_1) + a_1^\gamma, A_2^\gamma(x_2) + B_2^\gamma(x_1) + a_2^\gamma, A_3^\gamma(x_3) + B_3^\gamma(x_1, x_2) + a_3^\gamma),$$

where A_1^γ is an automorphism of $\mathbb{R}^{n_1}$, A_i^γ, $i = 2,3$ is an automorphism of $\mathbb{R}^{l_i}$, $i = 2,3$, $B_2^\gamma : \mathbb{R}^{n_1} \to \mathbb{R}^{l_1}$ is a linear map, and $B_3^\gamma : \mathbb{R}^{n_1} \oplus \mathbb{R}^{l_1} \to \mathbb{R}^{l_2}$ is a linear map.

If γ belongs to $\pi_1(F_2)$, then the restriction of γ to $I\!R^{n_1} \oplus I\!R^{l_1}$ is the identity and $B_3^\gamma = 0$, since we have supposed the fibration $(M_3, \nabla_{M_3}) \to (M_2, \nabla_{M_2})$ to be an affinely locally trivial affine fibration.

The holonomy of (M_2, ∇_{M_2}) is given by the action of $\pi_1(M_2)$ on $I\!R^{n_1} \oplus I\!R^{l_1}$ induced by the holonomy representation of (M_3, ∇_{M_3}). In fact the restriction of h_{M_3} to $I\!R^{n_1} \oplus I\!R^{l_1}$ factor through $\pi_1(M_2)$.

We deduce that the (M_2, ∇_{M_2}) holonomy's action of an element γ of $\pi_1(M_2)$, is

$$(A_1^\gamma(x) + a_1^\gamma, A_2^\gamma(y) + B_2^\gamma(x) + d_2^\gamma).$$

If γ is an element of $\pi_1(F_1)$, then $A_1^\gamma = id$, $a_1^\gamma = 0$ and $B_2^\gamma = 0$ since we have supposed that the fibration $(M_2, \nabla_{M_2}) \to (M_1, \nabla_{M_1})$ is an affinely locally trivial affine fibration.

Now, considering the holonomy of (M_2, ∇_{M_2}), we write the fact that $\pi_1(F_1)$ is normal in $\pi_1(M_2)$, then we obtain that the image of B_2^γ is contained in $H^0(\pi_1(F_1), I\!R^{l_1})$. Here the group $H^*(\pi_1(F), I\!R^{l_1})$, is the cohomology group of $\pi_1(F_1)$ related to its linear holonomy.

Considering the holonomy of (M_3, ∇_{M_3}), and writing that $\pi_1(M_2)$ is a normal subgroup of $\pi_1(M_3)$, we obtain that B_3^γ is contained in $H^0(\pi_1(F_2), I\!R^{l_2})$.

We have the representation

$$\pi_1(M_1) \longrightarrow Aff(F_1, \nabla_{F_1})/T_{F_1}$$

given by the first fibration. It leads to a flat bundle $\pi_{11}(M_1)$ over M_1 with typical fiber $Aff(F_1, \nabla_{F_1})/T_{F_1}$.

The bundle $(M_3, \nabla_{M_3}) \to (M_2, \nabla_{M_2})$, gives rise to a representation

$$\pi_1(M_2) \to Aff(F_2, \nabla_{F_2})/T_{F_2}$$

this leads to a representation

$$\pi_{F_1} : \pi_1(F_1) \to Aff(F_2, \nabla_{F_2})/T_{F_2}$$

and to a flat principal bundle $\pi_{12}(F_1)$ over F_1 with typical fiber $Aff(F_2, \nabla_{F_2})/T_{F_2}$ associated to π_{F_1}. Since $\pi_1(F_1)$ is a normal subgroup of $\pi_1(M_2)$, the action of $\pi_1(M_1)$ on this last bundle leads to a representation

$$\pi_1(M_1) \to Aut(\pi_{12}(F_1))$$

where $Aut(\pi_{12}(F_1))$ is the group of automorphisms of the bundle $\pi_{12}(F_1)$.

We will classify $3-$composition series of affine manifolds given:

The base space (M_1, ∇_{M_1}) and the fibers spaces (F_1, ∇_{F_1}), (F_2, ∇_{F_2}).

The representations $\pi_1(M_1) \to Aff(F_1, \nabla_{F_1})/T_{F_1}$, $\pi_1(F_1) \to Aff(F_2, \nabla_{F_2})/T_{F_2}$, and the representation $\pi_1(M_1) \to Aut(\pi_{12}(F_1))$.

The main tool we will use to make the classification of 3 composition series is 2 gerbe, on this purpose, let recall some definitions

Definitions 1.1.

A category is a family of objects and for each pair of objects X, Y, a set of arrows $Hom(X, Y)$, which satisfy usual rules.

An 2 category is of a family of objects, and for each pair of objects X, Y the arrows is a category $C(X, Y)$ which satisfies usual rules see [Bry-Mc].

An 2 gerbe on a manifold M is a sheaf of 2 categories C on M which satisfies the following see [Bry-Mc], [Bre]:

(2.1)For every element z of M, there exists an open set U_z which contains z such that $C(U_z)$ is not empty.

(2.2)Let U be an open set, for any pair of object x and y, contained in $C(U)$, there is an open covering $(U_i)_{i\in I}$ of U such that, for any i, the set of arrows between the restriction of x and y to U_i is not empty.

(2.3) For any 1 arrow $f : x \to y$ in $C(U)$, there is an inverse $g : y \to x$ up to a 2 arrow.

(2.4)The two arrows are invertible.

We will associate to our problem a sheaf of 2 categories.

First we define the following sheaf of categories C_1 on M_1:

To every $1-$connected open set U of M_1 of affine chart, we associate the category of affinely locally trivial affine bundles over U with typical fiber (F_1, ∇_{F_1}) such that the induced $\pi_{11}(U)$ bundle, is the restriction of $\pi_{11}(M_1)$ to U.

We now define on M_1 the sheaf of 2 categories C_2.

For every open set U, consider an element e of $C_1(U)$. It is an affinely locally trivial affine bundle over U with typical fiber (F_1, ∇_{F_1}). We associate to e the sheaf of category $C_2(e)$ which objects are a.l.t bundles over e with typical fiber (F_2, ∇_{F_2}) such that on e, the flat bundle with typical fiber $Aff(F_2, \nabla_{F_2})/T_{F_2}$ induced, is the one induced by the representation $\pi_1(F_1) \to Aff(F_2, \nabla_{F_2})/T_{F_2}$. Gluing conditions for the 2 sheaf C_2 are done using the representation $\pi_1(M_1) \to Aut(\pi_{12}(F_1))$.

This enables to associate to the open set U, $C_2(U) := \cup C_2(e), e \in C_1(U)$.

We will see that C_2 is in fact a 2 gerbe. An important fact is that this 2 gerbe is defined recursively, that is, we have first define the gerbe C_1. This will considerably simplify the expression of the classifying 3 cocycle which in this case will be an usual Cech 3 cocycle.

Let's precise now what 1 and 2 arrows are in the 2 category C_2.

Let e_1 and e_2 be objects of $C_1(U)$ where U is a simply connected open set of affine chart. A map between e_1 and e_2 can be represented by an affine map $t : U \to T_{F_1}$ acting on $U \times (F_1, \nabla_{F_1})$ as follows:

$$U \times (F_1, \nabla_{F_1}) \longrightarrow U \times (F_1, \nabla_{F_1})$$
$$(x, y) \longmapsto (x, t(x)y).$$

The map t can be viewed as an automorphism of the site of open sets of $U \times F_1$, so the map t induces a functor between the sheaf of categories $C_2(e_1)$ and $C_2(e_2)$, such maps t are 1 arrows of our 2 category.

A 2 arrow of our 2 category $C_2(U)$ over the one arrow t can be represented as a family of affine arrows

$$t_{i_1} : U \times U_{i_1} \times (F_2, \nabla_{F_2}) \longrightarrow U \times t(U_{i_1}) \times (F_2, \nabla_{F_2})$$

$$(x, y, z) \longmapsto (x, t(x)y, t_1(x, y)z),$$

where (U_{i_1}) is an open covering of (F_1, ∇_{F_1}) by one connected open set of affine charts, and $t_1 : U \times U_{i_1} \to T_{F_2}$ is an affine map.

It is easy to see that our category satisfy the axioms which defines 2 gerbes, so we have:

Proposition 1.2. *The* $2-$ *sheaf* C_2 *is a* 2 *gerbe.*

The classifying three cocycle.

The representation $\pi_1(M_1) \to Aff(F_1, \nabla_{F_1})/T_{F_1}$ induces an affine flat bundle V_1 over (M_1, ∇_{M_1}) with typical fiber T_{F_1}, we will call S_1 the sheaf of affine sections of this bundle. The representation $\pi_1(F_1) \to Aff(F_2, \nabla_{F_2})/T_{F_2}$ defines a flat bundle V_{12} over (F_1, ∇_{F_1}) with typical fiber T_{F_2}; we will call S_{12} the sheaf of affine sections of this bundle.

The representation $\pi_1(M_1) \to Aut(\pi_{12})(F_1)$ induces the sheaf S_{123} of affine maps $U \to S_{12}$, where U is an open set of M_1. This sheaf is a locally constant sheaf over M_1.

Now we consider an open covering $(U_i)_{i\in I}$ of (M_1, ∇_{M_1}) by 1 connected affine charts, For each i the 2 category $C_2(U_i)$ is not empty. We will choose in each U_i an element e_i of $C_1(U_i)$. Let i, j such that $U_i \cap U_j$ is not empty. The restriction of e_i and e_j to $U_i \cap U_j$ gives rise to an arrow

$\phi_{ij} : e_{i|U_i \cap U_j} \to e_{j|U_i \cap U_j}$.

This arrow can be expressed as a map

$$U_i \cap U_j \times (F_1, \nabla_{F_1}) \longrightarrow U_i \cap U_j \times (F_1, \nabla_{F_1})$$

$$(x, y) \longmapsto (x, t_{ij}(x)y).$$

Recall that the map ϕ_{ij} can also be viewed as a functor $C_2(e_i)_{|U_i \cap U_j} \to C_2(e_j)_{|U_i \cap U_j}$.

Now we consider the restriction of the functor $\phi_{ij} \circ \phi_{jk} \circ \phi_{ik}^{-1} = \psi_{ijk}$ to the family of $U_i \cap U_j \cap U_k$.

It can be represented by a family of maps

$$U_i \cap U_j \cap U_k \times U_{i_1} \times (F_2, \nabla_{F_2}) \longrightarrow U_i \cap U_j \cap U_k \times t_{ij} t_{jk} t_{ik}^{-1}(U_{i_1}) \times (F_2, \nabla_{F_2})$$

$$(x, y, z) \longmapsto (x, t_{ij}(x)t_{jk}(x)t_{ki}(x)(y), u_{ijk}(x, y)(z)),$$

where U_{i_1} is a one connected open set of affine chart of (F_1, ∇_{F_1}), $u_{ijk} : U_i \cap U_j \cap U_k \times U_{i_1} \to T_{F_2}$ is an affine map.

Now, we can restrict ψ_{ijk} to $U_i \cap U_j \cap U_k \cap U_l = U_{ijkl}$, writing the boundary of the chain ψ_{ijk}, we obtain $\rho_{ijkl} = \psi_{jkl}\psi_{ikl}^{-1}\psi_{ijl}\psi_{ijk}^{-1}$, ρ_{ijkl} can be viewed as a map

$$U_{ijkl} \times U_{i_1} \times (F_2, \nabla_{F_2}) \longrightarrow U_{ijkl} \times U_{i_1} \times (F_2, \nabla_{F_2})$$

$$(x, y, z) \longmapsto (x, y, w_{ijkl}(x, y)z),$$

as the family of map $t_{ij}t_{jk}t_{ki}$ defines a 2 Cech cocycle of S_1. The family ρ_{ijkl} can be viewed as sections of the bundle S_{123}.

Theorem 1.3. *The family of maps ρ_{ijkl} that we have just define is a 3 Cech cocycle.*

Proof.

We must calculate the boundary of the family of ρ_{ijkl}.

Let U_{ijklm} be the intersection $U_i \cap U_j \cap U_k \cap U_l \cap U_m$, we have:

$d(\rho_{ijklm}) =$

$\rho_{jklm} - \rho_{iklm} + \rho_{ijml} - \rho_{ijkm} + \rho_{ijkl}$

$= \psi_{klm}\psi_{jlm}^{-1}\psi_{jkm}\psi_{jkl}^{-1}$

$-(\psi_{klm}\psi_{ilm}^{-1}\psi_{ikm}\psi_{ikl}^{-1})$

$+\psi_{jml}\psi_{iml}^{-1}\psi_{ijl}\psi_{ijm}^{-1}$

$-(\psi_{jkm}\psi_{ikm}^{-1}\psi_{ijm}\psi_{ijk}^{-1})$

$+\psi_{jkl}\psi_{ikl}^{-1}\psi_{ijl}\psi_{ijk}^{-1} = 0.$

The associated 3 cocycle ρ_{ijkl} is not the obstruction to the existence of a composition serie, suppose that it vanishes.

This means that there is a family of maps

$$h_{ijk} : U_{ijk} \times U_{i_1} \times (F_2, \nabla_{F_2}) \longrightarrow U_{ijk} \times \psi_{ijk}(U_{i_1}) \times (F_2, \nabla_{F_2})$$

$$(x, y, z) \longmapsto (x, \psi_{ijk}(x)y, h'_{ijk}(x, y)z),$$

(where the map $h'_{ijk} : U_{ijk} \times U_{i_1} \to T_{F_2}$ is an affine map which boundary is ρ_{ijkl}) which is a 2 cocycle of $S_1 \oplus S_{123}$.

We have:

Theorem 1.4. *If the cocycle ρ_{ijkl} is trivial, then two cocycle h_{ijk} that we have just define is the obstruction to the existence of a composition serie associated to the bundles S_1, S_{12} and S_{123}.*

Proof.

If the cocycle h_{ijk} is trivial, then there exists a family of maps $b_{ij} : U_i \cap U_j \longrightarrow S_1 \oplus S_{123}$ such that the family of map $(t_{ij} + b_{ij})$ define a 1 Cech cocycle. This implies that the family of maps t_{ij} is a 1 cocycle up to a 1 boundary, then it defines an affine bundle over (M_1, ∇_{M_1}) with typical fiber (F_1, ∇_{F_1}) associated to π_{11}. Let (M_2, ∇_{M_2}) be its total space. The obstruction of the existence of an affine bundle over (M_2, ∇_{M_2}) with typical fiber (F_2, ∇_{F_2}) associated to S_1, S_{12} and S_{123} is given by h_{ijk}, since the open set $U \times U_{i_1}$ used to build the obstruction ρ can be viewed as open subsets of M_2.

Remark.

When the obstruction ρ vanishes, the cocycle h_{ijk} define on M_1 a gerbe which can be viewed as trivial 2 gerbe.

2. The general case.

In this part, we will classify composition series $(M_n, \nabla_{M_n}) \to ... \to (M_2, \nabla_{M_2}) \to (M_1, \nabla_{M_1})$.

We will denote by (F_i, ∇_{F_i}) the fiber of the a.l.t affine bundle $f_i : (M_{i+1}, \nabla_{M_{i+1}}) \to (M_i, \nabla_{M_i})$.

Let $i < j \leq n$, the map $f_{ij} = f_{j-1} \circ f_{j-2} \circ ... \circ f_i$, is an affine map $f_{ij} : (M_j, \nabla_{M_j}) \to (M_i, \nabla_{M_i})$. Since this map is a submersion and M_j is compact, we deduce that (M_j, ∇_{M_j}) is the total space of a locally trivial differentiable fibration over (M_i, ∇_{M_i}). The Serre bundle theorem implies the following exact sequence

$$1 \to \pi_1(F_i) \to \pi_1(M_{i+1}) \to \pi_1(M_i) \to 1$$

when $j = i + 1$.

The affine map f_{ij} define an affine bundle which is not necessarily an affinely locally trivial affine bundle.

Recall that the map f_i gives rise to a representation $\pi_i : \pi_1(M_i) \to Aff(F_i, \nabla_{F_i})/T_{F_i}$, and to a flat bundle V_i over M_i with typical T_{F_i}.

Thus we have a flat bundle V_{n-1} over M_{n-1} with typical fiber $T_{F_{n-1}}$. The group $\pi_1(F_{n-2})$ is a subgroup of $\pi_1(M_{n-1})$, thus we have a flat bundle V_{n-1n-2} over F_{n-2} with typical fiber $T_{F_{n-1}}$ induced by V_{n-1}.

The fundamental group $\pi_1(M_{n-2})$ of M_{n-2}, acts on V_{n-1n-2} via the representation π_{n-1}. This action defines a flat V'_{n-1n-2} over M_{n-2} with typical fiber V_{n-1n-2}. This bundle also gives rise to a bundle $V_{n-1n-2n-3}$ over F_{n-3} as $\pi_1(F_{n-3})$ is a subgroup of $\pi_1(M_{n-2})$. Recursively, we can define bundle $V_{n-1...n-i}$ over $n-i$ with typical fiber $V_{n-1...n-i+1}$ over F_i.

We can also define the representation π_{n-1n-2} which is the restriction of π_{n-1} to $\pi_1(F_{n-2})$. This representation induces on F_{n-2} a flat bundle π'_{n-1n-2} with typical fiber $Aff(F_{n-1}, \nabla_{F_{n-1}})/T_{F_{n-1}}$, $\pi_1(M_{n-2})$ acts on this bundle via π_{n-1}, one deduces a flat bundle over M_{n-2} with typical fiber π'_{n-1n-2} which induce a flat bundle $\pi'_{n-1n-2n-3}$ over F_{n-3}. Recursively, we can define bundle $\pi'_{n-1...1}$.

Remark also that considering the composition serie $(M_j, \nabla_{M_j}) \to .... \to (M_1, \nabla_{M_1})$ for $j < n$, one can also define the bundle $V_{jj-1....i}$ with typical fiber $V_{j...i+1}$ and base space F_i.

Let Let S_{n-1n-2} be the sheaf of affine sections of V_{n-1n-2} one may define the sheaf $S_{n-1n-2n-3}$ of affine sections of S_{n-1n-2} over F_{n-3}. Recursively, we can also define the sheaf $S_{n-1...1}$ of affine sections of $S_{n-1...2}$ over M_1. The gluing conditions for those sheaves are given by the bundles $\pi'_{n-1...i}$. One can also define in the same way the bundles $S_{i...1}$, $i \leq 1$.

Remark.

The bundles $V_{j\ldots1}$ and $S_{j\ldots1}$ depend only of the affine structures of (F_1,∇_{F_1}),...,(F_j,∇_{F_j}) and (M_1,∇_{M_1}). They can be defined without suppose the existence of a composition serie.

The classifying $n-$cocycle.

Given bundles $V_{ii-1\ldots1}$, and $S_{i\ldots1}$ $1\leq i\leq n-1$ as above,

We want to classify all composition series $(M_n,\nabla_{M_n})\rightarrow\ldots\rightarrow(M_1,\nabla_{M_1})$ associated to the family of bundles $V_{i\ldots1}$. To make this classification, we are first going to define an $n-$cocycle.

First we consider the trivialization of the flat bundle over M_1 with typical fiber $Aff(F_1,\nabla_{F_1})/T_{F_1}$ over M_1 induced by π_1.

It is defined by

$$U_i\cap U_j\times Aff(F_1,\nabla_{F_1})/T_{F_1}\longrightarrow U_i\cap U_j\times Aff(F_1,\nabla_{F_1})/T_{F_1}$$

$$(x,y)\longmapsto(x,\bar{t}_{ij}(y))$$

Consider for each x in $U_i\cap U_j$ an element $t_{ij}(x)$ of $Aff(F_1,\nabla_{F_1})$ over $\bar{t}_{ij}$ which depends affinely of x, one may define the 2 cocycle

$$t_{ijk}:U_i\cap U_j\cap U_k\longrightarrow T_{F_1}$$

$$x\longmapsto t_{ij}(x)t_{jk}(x)t_{ki}(x).$$

The family of t_{ijk} may be considered as local sections of the bundle S_1. It is the cocycle associated to the gerbe which at U_i associated the category of a.l.t affine bundles over U_i with typical fiber (F_1,∇_{F_1}), such that the bundle over U_i with typical fiber T_{F_1} associated is the restriction of V_1.

The map t_{ijk} induces a functor on the category of open sets of $U_i\cap U_j\cap U_k\times(F_1,\nabla_{F_1})$.

Consider a covering U_{i_1} of (F_1,∇_{F_1}) by affine 1 connected affine charts.

Let $U_{ijk}=U_i\cap U_j\cap U_k$. We associate to $U_{ijk}\times U_1$ the category of a.l.t affine bundles with typical fiber (F_2,∇_{F_2}) such that the bundle with typical fiber T_{F_2} associated is induced V_2.

On $U_{ijkl}=U_i\cap U_j\cap U_k\cap U_l$, the boundary of $t_{ij}t_{jk}t_{ki}$ gives rise to the map

$$v_{ijkl}:U_{ijkl}\times U_{i_1}\times(F_2,\nabla_{F_2})\longrightarrow U_{ijkl}\times U_{i_1}\times(F_2,\nabla_{F_2})$$

$$(x,x_1,x_2)\longmapsto(x,x_1,u_{ijkl}(x,x_1)(x_2)),$$

where the map u_{ijkl} are affine sections of the bundle S_{321}. The family of maps u_{ijkl} is a 3 cocycle.

Let $U_{1\ldots i}=U_1\cap U_2....\cap U_i$. For each j, we will consider an open covering U_{i_j} of (F_j,∇_{F_j}) by $1-$ connected open sets of affine charts.

Suppose that we have defined a family of maps

$$v_{1\ldots j}:U_{1\ldots j}\times U_{i_1}\times\ldots\times U_{i_{j-3}}\times(F_{j-2},\nabla_{F_{j-2}})\longrightarrow U_{1\ldots j}\times U_{i_1}\times\ldots\times U_{i_{j-3}}\times(F_{j-2},\nabla_{F_{j-2}})$$

$$(x, x_1, ..., x_{j-2}) \longmapsto (x, x_1, ..., x_{j-3}, u_{1...j-3}(x, x_1, ...x_{j-3})(x_{j-2}))$$

(where the family of maps $u_{1...j-3}$ are local affine sections of the bundle $S_{j...1}$) which represent a $j-1$ Cech cocycle. Then on $U_{1...j} \times U_{i_1} \times ... \times (F_{j-2}, \nabla_{F_{j-2}})$, one can define the sheaf of category C_j such that the objects $C_j(U_{1...j} \times ... \times U_{i_{j-2}})$ are affinely locally trivial affine bundles with typical fiber $(F_{j-1}, \nabla_{F_{j-1}})$ over $U_{1...j} \times U_{i_1} \times ... \times (F_{j-2}, \nabla_{F_{j-2}})$, and the canonical flat vector bundle with typical fiber $T_{F_{j-1}}$ associated is the restriction of V_{j-1}.

The map $v_{1...j}$ induces a functor $w_{1...j}$ in the category $C_j(U_{i...1} \times ... \times U_{i_{j-2}})$.

The restriction of the composition $w_{2...j+1} \circ ... \circ w^{(-1)^{k+1}}_{1...\hat{k}...j+1} \circ ... \circ w^{(-1)^{j+2}}_{1...j}$ on $U_{1...j+1} \times U_{i_1} \times ...Ui_{j-2}$ induces a map

$$v_{1...j+1} : U_{1...j+1} \times U_{i_1} ... \times U_{i_{j-2}} \times (F_{j-1}, \nabla_{F_{j-1}}) \longrightarrow U_{1...j+1} \times U_{i_1} ... \times U_{i_{j-2}} \times (F_{j-1}, \nabla_{F_{j-1}})$$

$$(x, x_1, ..., x_{j-2}) \longmapsto (x, x_1, ..., x_{j-2}, u_{1...j-2}(x, ..., x_{j-2})(x_{j-1}))$$

Proposition 2.1. The Cech chain $v_{1...j+1}$ that we have just define recursively is a j Cech cocycle.

Proof.

The proof will be made recursively. We have already verify the result if $j = 2$.

Suppose that the chain $v_{1...j}$ is a Cech cocycle for $k \leq j$, then the writing the boundary $d(v_{1...j+1})$ of $v_{1...j+1}$, we obtain

$$\sum_{l=1}^{l=j+2} (-1)^{l+1} v_{1...\hat{l}...j+2}$$

$=$

$$\sum_{l=1}^{j+2} (-1)^{l+1} (\sum_{m=1}^{m=l-1} w^{(-1)^{m+1}}_{1..\hat{m}..\hat{l}..j+2} \circ ... \circ w^{(-1)^l}_{1..\hat{l-1}\hat{l}...j+1} + \sum_{m=l+1}^{j+2} w^{(-1)^m}_{1..\hat{l}..\hat{m}..j+2}) = 0.$$

The cocycle $v_{1...n+1}$ is not the obstruction of the existence of a composition sequence associated to the family of bundles $S_{ii-1...1}$. If its cohomology class is zero, its means that there exists a chain $a_{1...n}$ which boundary is $v_{1...n+1}$. Suppose that the chain $z_{n-1} = a_{1...n} + v_{1...n}$ considered as an element of the Whitney sum of the bundles $S_{n-1...1} \oplus S_{n-2...1} = T_{n-1n-2}$ is an $n-1$ cocycle.

If the cohomology class of the cocycle z_{n-1} is zero, then it is the boundary of an $n-2$ cocycle $a_{1...n-1}$. We can define the chain $z_{n-2} = a_{1...n-1} + v_{1...n-1}$ considered as an element of $T_{n-1n-2n-3} = T_{n-1n-2} \oplus S_{n-3...1}$.

Suppose that we have define the cocycle z_{n-i} and its cohomology class is zero. It is the boundary of a chain $a_{1...n-i}$. This means that the chain $z_{n-(i+1)} = a_{1...n-i} + v_{1...n-i}$ is an $n-(i+1)$ cocycle viewed as an element of $T_{n-1...n-(i+2)} = T_{n-1...n-(i+1)} \oplus S_{n-(i+2)...1}$.

Theorem 2.2. *Suppose that we can construct a chain z_2 by the processus that we have just describe; then it is the obstruction to the existence of a composition serie associated to the family $S_{ii-1...1}$. When it vanishes the set of equivalence classes of composition series is given by $H^1(M_1, T_{n-1...1})$.*

Proof.

Suppose that we can define the cocycle z_2 and its cohomology class is zero, then then it is the boundary of an element z_1 of $T_{n-1...1}$.

This implies that up to a boundary the chain $t_{ij}t_{jk}t_{ki}$ is zero, we deduce that we can solve the first extension problem, there exists a bundle $(M_2, \nabla_{M_2}) \rightarrow (M_1, \nabla_{M_1})$ associated to S_1.

The obstruction of the existence of a composition serie $(M_3, \nabla_{M_3}) \rightarrow (M_2, \nabla_{M_2}) \rightarrow (M_1, \nabla_{M_1})$ is also given by the class of z_2 which is zero, so we can solve the second extension problem.

Suppose that we can solve the extension problem $(M_i, \nabla_{M_i}) \rightarrow ... \rightarrow (M_1, \nabla_{M_1})$ then obstruction to solve the extension problem $(M_{i+1}, \nabla_{M_{i+1}}) \rightarrow ... \rightarrow (M_1, \nabla_{M_1})$ is also given by the cohomology class of z_2 which is zero.

Recursively, we deduce that we can solve the extension problem $(M_n, \nabla_{M_n}) \rightarrow ... \rightarrow (M_1, \nabla_{M_1})$.

3. The Conceptualization.

The resolution of the extension problem, given the base space (M_1, ∇_{M_1}) and the bundle S_1 has been done using the gerbe theory. It is natural to think that the existence of a composition serie $(M_n, \nabla_{M_n}) \rightarrow ... \rightarrow (M_1, \nabla_{M_1})$ must be solve using $n-1-$gerbe theory. In this part, we are going to build a commutative $n-$gerbe theory.

On $n-$ Categories.

Supposed that the notion of i category is defined. An $i+1$ category C_{i+1}, is given by

The class of objects $O(C_{i+1})$,

i_1

the morphisms $Hom_{C_{i+1}}(x, y)$ between two objects x and y of C_{i+1} which is an i category,

i_2

For objects x, y, z in C_{i+1}, the composition i functor

$o_i : Hom_{C_{i+1}}(x, y) \times Hom_{C_{i+1}}(y, z) \rightarrow Hom_{C_{i+1}}(x, z)$

i_3

For each objects $x_1, ..., x_{i+4}$ in C_{i+1}, we will assume that the following strict condition: the composition

$$Hom_{C_{i+1}}(x_{i+3}, x_{i+4}) \times ... \times Hom_{C_{i+1}}(x_2, x_3) \times Hom_{C_{i+1}}(x_1, x_2) \rightarrow Hom_{C_{i+1}}(x_1, x_{i+4})$$

does not depend of the order in which it is made. More others conditions need to be specified, but we don't need them.

The notion of sheaf of i categories.

Now we define recursively the notion of sheaf of i categories on a topological space M.

We assume known the notion of sheaves of sets.

Supposed that we have already defined the notion of sheaves of i categories. An sheaf of $i+1$ categories C_{i+1} on the topological space M, will be a map which assign to each open set U a $i+1$ category $C_{i+1}(U)$, such that

For each inclusion $U \hookrightarrow V$, there exists a $i+1$ functor

$$c_{U,V} : C_{i+1}(V) \to C_{i+1}(U)$$

which satisfies $c_{U,V} \circ c_{V,W} = c_{U,W}$ for any open sets U, V and W such that $U \hookrightarrow V \hookrightarrow W$.

Gluing condition for objects. Consider an open covering (U_j) of M, and in each U_i an object A_j, we restrict each object We restrict each object of the family $(A_{j_1}, ..., A_{j_{i+3}})$ to $U_{j_1 ... j_{i+3}}$, we assume that if the composition

$$Hom_{C(U_{j_1 \cdots j_{i+3}})}(A_{j_{i+2}}, A_{j_{i+3}}) \times ... \times Hom_{C(U_{j_1 \cdots j_{i+3}})}(A_{j_1}, A_{j_2}) \longrightarrow Hom_{C(U_{j_1 \cdots j_{i+3}})}(A_{j_1}, A_{j_{i+3}})$$

does not depend on the order in which it is made then there exists a global object A which restriction on each U_i is A_i.

Gluing condition for arrows. For each pair of global objects x and y, $Hom_{C_{i+1}(M)}(x, y)$ is a sheaf of i categories.

The notion of n gerbe.

Now, we will define the notion of n gerbe where $n \geq 2$.

Consider a topological space M, endowed with a sheaf of i categories C_i for each $i = 1, ..., n$ such that

g_1

For each open set U, the set of objects of the category $C_i(U)$ is the same for each $i = 1, ...n$.

g_2

For each $x \in M$, we suppose that there is a neighborhood U_x of x such that $C_i(U_x)$ is not empty.

g_3

The sheaf of categories category C_1 is a gerbe with lien the abelian sheaf T_1 over M.

g_4

The arrows between two objects x, y considered as elements of the 2 category $C_2(U)$ is a category $Hom_{C_2(U)}(x, y)$ which objects are elements of $Hom_{C_1(U)}(x, y)$ the arrows between the objects x and y considered as elements of $C_1(U)$. The composition $o_2 : Hom_{C_2(U)}(x, y) \times Hom_{C_2(U)}(y, z) \to Hom_{C_2(U)}(x, z)$ transforms two 1 arrows f and g to gf where the product gf is considered in respect to the one of $Hom_{C_1(U)}(x, y) \times Hom_{C_1(U)}(y, z) \to Hom_{C_1(U)}(x, z)$.

We recall that for each x and y in $C_2(U)$ the 1 arrows between x and y are the objets of the category $Hom_{C_2(U)}(x, y)$, we have just precise how the functor

o_2 acts on objects, not how it acts on maps. We deduce that the product is known up to 2 arrows.

The 1 arrows between x and y considered as elements of $C_3(U)$ is $Hom_{C_1(U)}(x,y)$, the 2 arrows are the 2 arrows of $Hom_{C_2(U)}(x,y)$.

The product $o_3 : Hom_{C_3(U)}(x,y) \times Hom_{C_3(U)}(y,z) \to Hom_{C_3(U)}(x,z)$ transform two 1 arrows f and g onto gf, the product is considered in respect to the one of $C_1(U)$ and two 2 arrows h and k in kh the product, considered is the one of $C_2(U)$. We can also say that the product is known up to 3 arrows.

Recursively, suppose that we have defined the $1, ..., i$ arrows of the category $C_i(U)$, then the 1 arrows of $C_{i+1}(U)$ are the arrows of $C_1(U)$,..., the i arrows of the category $C_{i+1}(U)$ is the i arrows of the category $C_i(U)$.

g_5

Suppose also that we have define recursively the product of $l \leq i$ arrows of $Hom_{C_i(U)}(x,y)$. Then the product $o_{i+1} : Hom_{C_{i+1}(U)}(x,y) \times Hom_{C_{i+1}(U)}(y,z) \to Hom_{C_{i+1}(U)}(x,z)$ is an i functor which send two l arrows $l \leq i$ in $C_{i+1}(U)$ onto the one with respect to the composition in $C_l(U)$. We can also remark that the product o_{i+1} is defined up to $i+1$ arrows.

g_6

We suppose that in $C_1(U)$ the arrows are invertible, in $C_2(U)$, a 1 arrow is invertible up to a 2 arrow, a 2 arrow is invertible, in $C_i(U)$, a 1 arrow is invertible up to a 2 arrow, a 2 arrow is invertible up to a 3 arrow,... a $i-1$ arrow is invertible up to a i arrow, and i arrow are invertible.

g_7

Given an object f of $Hom_{C_2}(x,x)$, where x is an object of $C_2(U)$, the set of morphisms of f is isomorphic to $T_2(U)$ where T_2 is a sheaf over M. More generally, let g be an $i-1$ map in the category $C_i(U)$, the set of morphisms of g is isomorphic to $T_i(U)$ where T_i is a sheaf over M.

g_8

Any two objets of $C_i(U)$ $i \leq n$ are locally isomorphic.

Definition 3.1.

A family of sheaves of i categories C_i for each $i \leq n$ which satisfy the conditions $g_1, ..., g_8$, will be called an $n-$gerbe.

The classifying $n+1-$cocycle.

In this part, we are going to consider n gerbes such that for each i, the sheaf T_i is a commutative sheaf.

We are going to associate to each $n-$gerbe a classifying $n+1-$Cech cocycle which takes values in T_n.

We will assume that M is a manifold, U_i is an open covering of M such that $C_i(U)$ is not empty and for each family $\{i_1, ..., i_k\}$ we set $U_{i_1 ... i_k} = U_{i_1} \cap U_{i_2} ... \cap U_{i_k}$.

We first choose in each open subset U_i an object x_i, we may consider x_i as an object of $C_1(U_i)$. If U_{ij} is not empty, then there is an arrow of t^1_{ij} between

x_i and x_j (which is an isomorphism). we can write the two chains

$$c_2(U_{i_1 i_2 i_3}) = t^1_{i_1 i_2} t^1_{i_2 i_3} t^1_{i_3 i_1}.$$

As C_1 is a gerbe, this implies that the Cech boundary of c_2 is trivial.

But one may also consider arrows of C_1 as 1 arrows of C_2, this implies that the $d(c_2)$ is trivial up two a $3-$ T_2 chain $c_3(U_{i_1 i_2 i_3 i_4})$ represented by a 3 arrows. It results from i_3 that c_3 is a $3-$ Cech cocycle.

Suppose that we have defined recursively a Cech $j+1-$cocycle associated to the sheaf of $j-$categories C_j. It is a family of elements $c_{j+1}(U_{i_1 \dots i_{j+2}})$ of j arrows of $C_j(U_{i_1 \dots i_{j+2}})$. As the boundary of c_{j+1} is trivial, it may be represented by an $j+2$ Cech chain c_{j+2} of T_{j+1} which is a $j+2$ cocycle. So we can deduce recursively the existence of an $n+1$ T_n cocycle c_{n+1} associated to the $n-$ gerbe.

Definitions 3.2.

An $n-$gerbe, will be said $n-$trivial if the cohomology class of the associated $n+1-$cocycle c_{n+1} that we have just define is trivial.

Let C, and C' two $n-$gerbes associated to the family of sheaves T_1,..., T_n, which are locally isomorphic i.e, each x in M, has an open neighborhood U_x such that $C_i(U_x)$ and $C'_i(U_x)$ are not empty, and objects of $C_i(U_x)$ and $C'_i(U_x)$ are locally isomorphic. We will say that the locally isomorphic $n-$gerbes C and C' are equivalent if and only if for every open set U of M, there is a n isomorphism $\phi_n(U) : C_n(U) \rightarrow C'_n(U)$ such that the restrictions of $\phi_n(U)$ and $\phi_n(V)$ on $U \cap V$ coincide with $\phi_n(U \cap V)$.

Suppose that the class $[c_{n+1}]$ of the cocycle c_{n+1} is trivial. It implies that it is the boundary of a chain a_n. Let consider the chain $z_n = a_n + c_n$ of $T_{n-1} \oplus T_n$, if its cohomology class is trivial, it implies that it is the boundary of a chain a_{n-1}, we set $z_{n-1} = a_{n-1} + c_{n-1}$. Suppose that recursively we have defined the chain z_{n-i}.

The n gerbe is said $n-i$ trivial, if we can define the cocycle z_{n-i} by the processus above. The fact that the n gerbe is $n-i$ trivial means that it can be considered as a $n-i-1$ gerbe. as follows: we consider the $n-i-1$ gerbe C' such that for every open set U, $C'_j(U) = C_j(U)$ if $j < n-i-1$. We define the $n-i-1$ maps of $C_{n-i-1}(U)$ to be $T_{n-i-1}(U) \oplus \dots \oplus T_n(U)$.

Proposition 3.3.

The set of equivalence classes of locally isomorphic n trivial gerbes is given by $H^{n+1}(M, T_n)$.

Proof.

We have assigns to every $n-$gerbe a $n+1$ cocycle c_{n+1} if it class is trivial, it is equivalent to saying that the $n-$gerbe is trivial. This implies that the map $C \rightarrow c_{n+1}$ is injective.

On the other hand, given a cocycle c_{n+1}, we can find a family of cocycles c_2,...,c_n, such that c_{i+1} is induced by c_i by the processus described to build the classifying cocycle, and c_n induces c_{n+1}.

We can extend our definition and defines $\infty-$gerbe.

Definition 3.4.
The family C_n $n \in I\!N$ of $n-$sheaves of categories over the manifold M is an $\infty-$gerbe C if and only if for each $n \in I\!N$, the family C_1,...,C_n is an $n-$gerbe and the family of sheaves T_n, $n \in I\!N$ is an inductive system such that the map $i_n : T_n \rightarrow T_{n+1}$ sends c_{n+1} onto c_{n+2}, where c_{n+1} is the classifying cocycle associated to the gerbe C_1,... C_n. We will call the inductive limit of c_n the classifying cocyle.

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